

Parametric estimation in noisy blind deconvolution model: a new estimation procedure

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Abstract: In a parametric framework, the paper is devoted to the study of a new estimation procedure for the inverse filter and the level noise in a complex noisy blind discrete deconvolution model. Our estimation method is a consequence of the sharp exploitation of the specific properties of the Hankel forms. The distribution of the input signal is also estimated. The strong consistency and the asymptotic distribution of all estimates are established. A consistent simulation study is added in order to demonstrate empirically the computational performance of our estimation procedures.

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1. Introduction

Let $(Y_t)_{t \in \mathbb{Z}}$ be the output process of an unknown deterministic linear time-invariant sequence $(u_t)_{t \in \mathbb{Z}}$, which is driven by an unobservable input sequence $(X_t)_{t \in \mathbb{Z}}$ added with a noise sequence $(\sigma_0 W(t))_{t \in \mathbb{Z}}$, where σ_0 is an unknown level noise. In other words, $(Y_t)_{t \in \mathbb{Z}}$ is issued of the noisy blind deconvolution model defined by

$$Y_t = (u \star X)_t + \sigma_0 W_t = \sum_{k \in \mathbb{Z}} u_k X_{t-k} + \sigma_0 W_t, \quad \forall t \in \mathbb{Z} \quad (1.1)$$

where $(X_t)_{t \in \mathbb{Z}}$ is assumed to be a complex discrete finite valued input process and the real-valued filter $(u_t)_{t \in \mathbb{Z}} \in l_1(\mathbb{Z})$ is supposed to invertible.

Since the sequence $(u_t)_{t \in \mathbb{Z}}$ is invertible, it exists $\theta = (\theta_t)_{t \in \mathbb{Z}}$ the inverse filter of $(u_t)_{t \in \mathbb{Z}}$, such that $\sum_{t \in \mathbb{Z}} u_t \theta_{k-t} = \mathbb{1}_{\{k=0\}}$. Note that if $(u_t)_{t \in \mathbb{Z}}$ has a finite length, the system is a noisy moving average and if $(\theta_t)_{t \in \mathbb{Z}}$ has a finite length, the system is a noisy autoregressive. From n observations $(Y_t)_{\{1 \leq t \leq n\}}$, the objective is to reconstitute the distribution of the input process $(X_t)_{t \in \mathbb{Z}}$ which requires the estimation of the level of noise σ_0 and the filter $(u_t)_{t \in \mathbb{Z}}$. In digital signal framework, that is, when $(X_t)_{t \in \mathbb{Z}}$ is a discrete valued input, the

Bayesian theory combined with the MCMC methods is often used to obtain the a posteriori distribution of the signal process $(X_t)_{t \in \mathbb{Z}}$ (Liu & Chen [1995]; Li & Shedden [2001]). Here, we adopt the approach which consists in estimating the inverse filter $(\theta)_{t \in \mathbb{Z}}$ instead of estimating the filter itself $(u)_{t \in \mathbb{Z}}$. The problem of estimating $(\theta_t)_{t \in \mathbb{Z}}$ as well as the distribution of the input process from the observed output data $(Y_t)_{t=1, \dots, n}$ is known as an identification problem. This identification problem when the distribution of the input signal is discrete with a finite number of possible values, is discussed in a number of papers: Li [1992, 1993, 1995, 1999, 2003], Li & Mbarek [1997], Gamboa & Gassiat [1996] or Gassiat & Gautherat [1998, 1999] and Gautherat [1997, 2002]. In the real and complex cases without noise, Li [1995] proposed an estimation method for the inverse filter when the support points of the input signal are supposed to be known. Gamboa & Gassiat [1996] in the non-noisy real case under a general setting (unknown distribution of the input signal) proved the consistency of the inverse filter estimate and that of the cardinality of the support's points. Gassiat & Gautherat [1998] extended in some sense the previous paper in considering data with an additive noise and proposed consistent estimates for the support's points and also for the level noise. Gassiat & Gautherat [1999] studied the rate of convergence of the signal estimate and the inverse filter estimate in the parametric framework whereas Gautherat [2002] also established asymptotic distribution for the points and their corresponding mass of the signal distribution. To construct the cost function, some of these authors explicitly used the alphabet of the signal values (Li [2003, 1992, 1993, 1995, 1999]; Li & Mbarek [1997]), whereas Gamboa & Gassiat [1996], Gassiat & Gautherat [1998, 1999] and Gautherat [1997, 2002] used only the cardinality of the alphabet (non communicative situation).

Chen & Li [1995], Gamboa & Gassiat [1996, 1997a, 1997b], and Gunther & Swindlehurst [2000] showed that incorporating the finite alphabet information into blind deconvolution procedures can greatly improve the accuracy of the filter estimation and that of the signal distribution in the non-noisy situation. Due to the judicious utilization of the finite alphabet information, these methods enjoyed a number of desirable properties in the non-noisy situation such that, the ability to handle (with super statistical efficiency) a large class of filters $(u_t)_{t \in \mathbb{Z}}$, the ability to handle non-stationary (see the papers of Li) or non i.i.d. signals (Gamboa & Gassiat [1996]) without modeling or estimating their statistical characteristics.

In the model (1.1), the blind deconvolution of the data is much more complicated due to the presence of the noise. Gassiat and Gautherat [1998, 1999] and Gautherat [1997, 2002] proposed a consistent estimation procedure which is based on the minimization of a penalized empirical contrast function. In practice, this method needs to adjust the penalty term and requires a starting point which is near enough to the true value in order to avoid a local minimizer.

Among these works, the main contribution of our paper is to provide in the complex case, a new estimation procedure of the level of noise and the inverse filter, which is a consequence of the sharp study of the Hankel matrix when the noise is supposed to be Gaussian. Our estimation procedure is built on an ex-

explicit empirical criteria (it is not the same in the papers of Gassiat & Gautherat [1998, 1999] and Gautherat [1997, 2002]), and it is based on the roots of an empirical function. Our estimation method is competitive from a theoretical point of view (consistency and asymptotic distribution of all estimates) and from a practical side (our numerical results are quite good).

The paper is organized as follows. The assumptions on the model defined by (1.1) are given in Section 2. In the same section, the level of noise, the inverse filter and the law of the input signal are characterized. These characterizations are used to define in Section 3 our estimation procedures. The strong consistency and the asymptotic distribution of all the estimates are stated in Section 4. The proofs are postponed in Section 7. Section 5 deals with a simulation study in which the computational performance of our estimation procedures is empirically demonstrated. Some concluding remarks including a comparison of our numerical results with those given in Gassiat & Gautherat [1998], are made in Section 6.

2. Assumptions and Characterization

2.1. Assumptions

- (M1) $(X_k)_{k \in \mathbb{Z}}$ is a sequence of discrete complex random variables with a common discrete support $a = (a_1, \dots, a_p) \in \mathbb{C}^p$; a is unknown, $a_i \neq a_j$ for $i \neq j$. The integer $p \geq 2$ is known. The components $(a_j)_{1 \leq j \leq p}$ of a are given by the lexicographical order.
- (M1b) $(X_k)_{k \in \mathbb{Z}}$ is identically distributed with support points $a \in \mathbb{C}^p$ and $\Pi = (\pi_j)_{\{j=1, \dots, p\}}$ is such that $\mathbb{P}(X_k = a_j) = \pi_j \forall j \in \{1, \dots, p\}$ and for any $k \in \mathbb{Z}$. The integer $p \geq 2$ is known. The components $(a_j)_{1 \leq j \leq p}$ of a are given by the lexicographical order.
- (M2) $(X_k)_{k \in \mathbb{Z}}$ is a stationary ergodic process.
- (M3) $\forall n \in \mathbb{N}^*, \forall (j_1, \dots, j_n) \in \{1, \dots, p\}^n, \mathbb{P}(X_1 = a_{j_1}, \dots, X_n = a_{j_n}) > 0$.
- (M4) $\forall k \in \mathbb{Z}, W_k = W_k^R + iW_k^I$, where $W^R = (W_k^R)_{k \in \mathbb{Z}}$ and $W^I = (W_k^I)_{k \in \mathbb{Z}}$ are independent. The random variables $(W_k^R)_{k \in \mathbb{Z}}$ and $(W_k^I)_{k \in \mathbb{Z}}$ are real centered i.i.d. Gaussian with a variance equal to $(1/2)$ and they are both independent of (X_k) .
- (M5) $U(x) = \sum_k u_k e^{ikx}$ is continuous and does not vanish on $[0, 2\pi]$.

Note that Assumption (M5) guarantees that $(u_t)_{t \in \mathbb{Z}}$ is invertible and that both $(u_t)_{t \in \mathbb{Z}}$ and its inverse are in $l_1(\mathbb{Z})$. Since $(u_t)_{t \in \mathbb{Z}}$ is invertible, the initial observed process $(Y_t)_{t \in \mathbb{Z}}$ can be transformed by applying any filter s to Y_t . The resulting process $Z_t(s)$ is then defined by

$$\forall t \in \mathbb{Z}, Z_t(s) = (s \star Y)_t. \quad (2.1)$$

- (M6) The set of the filters is defined by $\Theta = \{s(\xi) \in l_1(\mathbb{Z}), \xi \in \mathcal{K}\}$, where the function s is known, $s \in \mathcal{C}^1$ is injective. The true inverse filter θ is in Θ . The set $\mathcal{K} \subset \mathbb{R}^d$, $d \in \mathbb{N}^*$ is compact such that if $\xi, \tilde{\xi} \in \mathcal{K}$ satisfy

$s_k(\xi) = rs_{k-l}(\tilde{\xi})$, $\forall k \in \mathbb{Z} \Rightarrow r = 1$, $l = 0$, and $\xi = \tilde{\xi}$. The parameter ξ is unknown.

Remark 2.1.

(i) Assumption **(M6)** describes the set Θ at which the filters we consider, belong. The set Θ is parametric via the unknown parameter vector ξ , so that estimating the inverse filter θ is reduced to estimate ξ_0 .

(ii) Since the true inverse filter θ belongs to Θ and since the function s is injective, it implies that it exists ξ_0 in the interior of \mathcal{K} such that $s(\xi_0) = \theta$. Moreover, the last part of Assumption **(M6)** guarantees the identifiability of the model; in particular, it allows to avoid problems of scale and delay.

(iii) In the non noisy case, note that $Z_t(\theta) = X_t$, $\forall t \in \mathbb{Z}$ where $Z_t(\cdot)$ is defined by equation (2.1). Our estimation procedures will be based on the process $Z_t(s)$, $s \in \Theta$.

2.2. Characterizations

In the non-noisy framework ($\sigma_0 = 0$), Gamboa & Gassiat [1996] stated that, under **(M1)**, **(M3)**, **(M4)** and **(M5)**, $Z_t(s)$ takes at most p distinct values if and only if $s = \theta$ up to scale and delay. The definition of Θ as a subset of $l_1(\mathbb{Z})$ avoids identifiability problems which could be generated by possible changes in scale or delay. But since the characterizations of σ_0 and θ are valid for any inverse filter in $l_1(\mathbb{Z})$, from now and only for this paragraph, let us consider any filter s in $l_1(\mathbb{Z})$.

The random variable $Z_t(s)$ for $s \neq \theta$ having at least p points of support, the characterization of σ_0 and Θ is made via a contrast function which is able to distinguish discrete random variables whose support is of cardinality p from others whose support is of cardinality greater than p .

Let $d(s)$ be the conjugate moment column vector of dimension $(p+1)^2$ defined by

$$d_{j(p+1)+(k+1)}(s) = \mathbb{E} \left((Z_0(s))^k \overline{(Z_0(s))^j} \right), \forall (j, k) \in \{0, \dots, p\}^2, \forall s \in l_1(\mathbb{Z}). \quad (2.2)$$

Note that if one rewrites $d(s)$ in a $((p+1) \times (p+1))$ -matrix $D(s)$ where j and k correspond respectively to the columns and the rows then $D(s)$ is a Hankel matrix. For all $s \in l_1(\mathbb{Z})$ and $\sigma \geq 0$, it is always possible to derive the conjugate pseudo-moment $\tilde{d}(\sigma, s)$ from $d(s)$ in inverting the following system:

$$d(s) = A(\sigma \|s\|_2) \tilde{d}(\sigma, s), \quad (2.3)$$

where $\|s\|_2$ denotes the l_2 -norm of s . The matrix $A(\sigma \|s\|_2)$, which depends on $\sigma \|s\|_2$, is an invertible $(p+1)^2 \times (p+1)^2$ -matrix such that for any pair $(j, k) \in \{0, \dots, p\} \times \{0, \dots, p\}$ and $\forall 0 \leq m \leq j$, $0 \leq l \leq k$, $\forall \beta \in \mathbb{R}$,

$$A_{j(p+1)+k+1, m(p+1)+l+1}(\beta) = C_k^l C_j^m \gamma_{j-m, k-l} \beta^{k-l+j-m},$$

where $\gamma_{m,l} = \mathbb{E}((W_0)^l (\overline{W_0})^m)$. Note that under **(M4)** one gets $\gamma_{m,l} = \mathbb{I}_{m=l} m!$. Since $\gamma_{m,l} \neq 0 \iff l = m$, denote $\gamma_{m,l} = \gamma_m$. The inversion of the system (2.3) is explicit and $A^{-1}(\sigma \|s\|_2)$ is defined by the following: $\forall (j, k) \in \{0, \dots, p\}^2$ such that $0 \leq m \leq (j - k \vee 0)$, and $\forall \beta \in \mathbb{R}_+$,

$$A_{j(p+1)+k+1, m(p+1)+k-j+m+1}^{-1}(\beta) = (-1)^{j-m} C_k^{j-m} C_j^m \gamma_{j-m} \beta^{2(j-m)}. \quad (2.4)$$

Remark 2.2. Denote by $(R(s))_t = (s \star u \star X)_t$ and $(V(s))_t = (s \star W)_t$, $\forall t \in \mathbb{Z}$ and $\forall s \in l_1(\mathbb{Z})$. Then, $\forall \sigma \leq \sigma_0$ and due to the infinite divisibility of the Gaussian distribution, $\forall (j, k) \in \{0, \dots, p\}^2$, $\tilde{d}_{j(p+1)+(k+1)}(\sigma, s)$ is equal to

$$\mathbb{E} \left[\left((R(s))_0 + \sqrt{\sigma_0^2 - \sigma^2} (V(s))_0 \right)^k \left((\overline{R(s)})_0 + \sqrt{\sigma_0^2 - \sigma^2} (\overline{V(s)})_0 \right)^j \right].$$

On the other hand, if $\sigma > \sigma_0$, $\tilde{d}(\sigma, s)$ has no explicit form, one does not know if it corresponds to a random variable moment. This explains why $\tilde{d}(\sigma, s)$ is called the pseudo-moment vector.

Next, transform the pseudo-moment vector $\tilde{d}(\sigma, s)$ in a $((p+1) \times (p+1))$ -matrix $\tilde{D}(\sigma, s)$ where j and k correspond respectively to the columns and the rows i.e. $\tilde{D}(\sigma, s) = \left(\tilde{d}_{j(p+1)+(k+1)}(\sigma, s) \right)_{(j,k) \in \{0, \dots, p\}^2}$. Then, let J be the real function defined by:

$$\forall \sigma \geq 0, \forall s \in l_1(\mathbb{Z}), \quad J(\sigma, s) = \det(\tilde{D}(\sigma, s)). \quad (2.5)$$

The characterizations of θ , σ_0 , a and Π in the model described by relation (1.1) or equivalently by relation (2.1), are made through the function J defined by (2.5). They are established in Gassiat & Gautherat [1999] for θ and σ_0 , and in Gautherat [1997, 2002] for a and Π under a more general setting:

- (i) Under assumptions **(M1)**, **(M3)**, **(M4)** and **(M5)**, the true level noise σ_0 and the true inverse filter $\theta = s(\xi_0)$ satisfy

$$\forall \sigma < \sigma_0, \quad J(\sigma, s(\xi)) > 0, \forall \xi \in \mathcal{K} \quad (2.6)$$

$$J(\sigma_0, s(\xi)) = 0 \quad \text{iff} \quad s(\xi) = \theta \text{ up to scale and delay.} \quad (2.7)$$

- (ii) Under **(M1)**, the distribution points $a = (a_i)_{i=1, \dots, p}$ are the roots of the polynomial function p_{v^*} in $\mathbb{C}[X]$ defined by $p_{v^*}(x) = \sum_{j=0}^p v_j^* x^j$, where v^* denotes the eigenvector associated with the smallest eigenvalue of the matrix $\tilde{D}(\sigma_0, \theta)$.
- (iii) Under **(M1b)**, the distribution $\Pi = (\pi_1, \dots, \pi_p)$ is uniquely determined as the solution of the following linear system in $(q_1, \dots, q_p) \in [0, 1]^p$: $\mathbb{E}(X_0^k) = \sum_{i=1}^p q_i a_i^k \quad \forall k = 0, \dots, p-1$.

Note that (i) implies that $\sigma_0 = \min\{\sigma > 0; \exists s \in l_1(\mathbb{Z}) : J(\sigma, s) = 0\}$.

3. Estimation procedures

To construct our estimates, we consider again the filters of the form $s(\xi)$, where $\xi \in \mathcal{K}$ is unknown and s is a known function. For any $\xi \in \mathcal{K}$, let us consider the truncated sequence $\bar{s}(\xi)$ of $s(\xi) \in \Theta \subset l_1(\mathbb{Z})$ as $\bar{s}_k(\xi) = s_k(\xi) \mathbb{1}_{|k| \leq k(n)} \forall k \in \mathbb{Z}$, where $k(n)$ is a sequence of nonnegative integers increasing with n . Denote also $\|s(\cdot)\|_{n,2} = \|\bar{s}(\cdot)\|_2$. Define the truncated version of relation (2.1) that is,

$$\forall \xi \in \mathcal{K}, \quad \forall t = 1 + k(n), \dots, n - k(n), \quad Z_t(\bar{s}(\xi)) = \sum_{k=-k(n)}^{k(n)} s_k(\xi) Y_{t-k}.$$

Denote $d_n(\xi)$ the empirical conjugate moment vector of dimension $(p+1)^2$, whose general term $d_{j(p+1)+k+1,n}(\xi)$ is defined as the empirical version of (2.2),

$$\forall \xi \in \mathcal{K}, \quad d_{j(p+1)+k+1,n}(\xi) = \frac{1}{n - 2k(n)} \sum_{t=1+k(n)}^{n-k(n)} Z_t^k(\bar{s}(\xi)) \overline{Z_t^j(\bar{s}(\xi))}.$$

Then, similarly to (2.3), define $\tilde{d}_n(\xi)$ as the empirical conjugate pseudo-moment vector whose general term $\tilde{d}_{j(p+1)+k+1,n}(\xi)$ is the solution of the following triangular system:

$$\forall \xi \in \mathcal{K}, \forall \sigma \geq 0, \quad \tilde{d}_n(\sigma, \xi) = A^{-1}(\sigma \|\bar{s}(\xi)\|_2) d_n(\xi), \quad (3.1)$$

where $A^{-1}(\beta)$ is the matrix defined by (2.4). Finally, let J_n be the empirical version of J defined by (2.5):

$$\forall \xi \in \mathcal{K}, \forall \sigma \geq 0, \quad J_n(\sigma, \xi) = \det(\tilde{D}_n(\sigma, \bar{s}(\xi))),$$

where \tilde{D}_n is the $((p+1) \times (p+1))$ -matrix corresponding to the rewriting of the empirical pseudo-moment vector $\tilde{d}_n(\sigma, \xi)$ in a matrix form. Finally, let us define all the estimates:

- $(\hat{\sigma}_0, \hat{\xi}_0)$ is the solution of the following system

$$\begin{cases} J_n(\hat{\sigma}_0, \hat{\xi}_0) = 0, \\ \hat{\sigma}_0 = \min \{ \sigma \in \mathbb{R}_+; \exists \xi \in \mathcal{K} : J_n(\sigma, \xi) = 0 \}. \end{cases}$$

- $\hat{\theta}$ is defined as $\hat{\theta} = \bar{s}(\hat{\xi}_0)$.
- The support points $(\hat{a}_1, \dots, \hat{a}_p)$ in \mathbb{C}^p are the roots rearranged by the lexicographic order of the polynomial function $p_{\hat{v}^*}(x) = \sum_{j=0}^p \hat{v}_j^* x^j$ in $\mathbb{C}[X]$, where $\hat{v}^* = (\hat{v}_0^*, \dots, \hat{v}_p^*)$ denotes the eigenvector associated with the smallest eigenvalue of the matrix $\tilde{D}_n(\hat{\sigma}_0, \bar{s}(\hat{\xi}_0))$.
- The probability vector $(\hat{\pi}_1, \dots, \hat{\pi}_p)$ is uniquely determined as the solution of the following linear system in $(q_1, \dots, q_p) \in [0, 1]^p$: $\tilde{d}_{k+1,n}(\hat{\sigma}_0, \bar{s}(\hat{\xi}_0)) = \sum_{i=1}^p q_i \hat{a}_i^j, \quad \forall j = 0, \dots, p-1$.

Existence of $(\hat{\sigma}_0, \hat{\xi}_0)$ for any $n \in \mathbb{N}^*$.

For any $n \in \mathbb{N}^*$, fix $\xi \in \mathcal{K}$ and observe $d_n(\xi)$. Then, $\tilde{d}_n(\sigma, \xi)$ is obtained via equation (3.1) and depends only on the unknown parameter σ throughout $\sigma \|\bar{s}(\xi)\|_2$ since $\bar{s}(\xi)$ is supposed to be fixed. For any $v \in \mathbb{C}_*^{p+1}$, consider the hermitian form

$$Q_v(\sigma) = \sum_{k=0}^p \sum_{j=0}^p \tilde{d}_{j(p+1)+(k+1),n}(\sigma, \xi) v_k \bar{v}_j.$$

Note that Q_v is a polynomial function in σ with real coefficients. For any $v \in \mathbb{C}_*^{p+1}$, note that $Q_v(0) = \text{Det}(D_n(\xi)) > 0$, where $D_n(\xi)$ is a Hankel matrix of dimension $((p+1) \times (p+1))$ which corresponds to the rewriting of the vector $d_n(\xi)$. Note also that the highest degree of Q_v is equal to $2p$.

Then, if p is an odd number, the coefficient of the highest term is equal to $(-1)^p p! (\|s(\xi)\|_{n,2})^{2p} v_p \bar{v}_p$ which is negative. It entails that it exists a real positive $\tilde{\sigma}_n$ such that $Q_v(\tilde{\sigma}_n) = 0$ that is, the hermitian form is degenerate and then $J_n(\tilde{\sigma}_n, \xi) = 0$.

On the other hand, if p is an even number, choose any $v = (v_0, v_1, \dots, v_p)' \in \mathbb{C}_*^{p+1}$ such that $v_p = 0$, where the v' denotes the transposed vector of v . Then, the term of the highest degree is negative and is of order $\sigma^{2(p-1)}$. As previously, it allows to conclude to the existence of a real positive $\tilde{\sigma}_n$ which is a zero for $J_n(\cdot, \xi)$.

4. Main results

Up to now, for any function F with one or more arguments, set $D^r F(y)$ be the value at y of the r -th differential of F and set $\partial_{i_1, \dots, i_l}^{r_{i_1}, \dots, r_{i_l}} F(y)$ be the value at y

of the $(\sum_{k=1}^l r_{i_k})$ -th partial derivative of F , where r_{i_k} is the order of the derivative

with respect to its i_k -th coordinate. Denote also $d(\xi)$ the $(p+1)^2$ -vector with components $d_{j(p+1)+(k+1),n}(\xi) := d_{j(p+1)+(k+1),n}(s(\xi))$ defined by relation (2.2). Some extra other assumptions are needed to establish the consistency and the asymptotic distribution of all estimates.

$$(M7) \quad k(n) = o(\sqrt{n}), \quad \sum_{|k| > k(n)} |\theta_k| = o\left(\frac{1}{\sqrt{n}}\right) \text{ as } n \rightarrow +\infty$$

$$(M8) \quad \sqrt{n} (d_n(\xi_0) - d(\xi_0), D^1 d_n(\xi_0) - D^1 d(\xi_0)) \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, \Gamma). \text{ Denote } \Gamma_1 \text{ the asymptotic variance of } \sqrt{n} (d_n(\xi_0) - d(\xi_0)).$$

(P) The application $\xi \mapsto s(\xi)$ is twice continuously differentiable. For any $i = 1, \dots, d$, $(\partial_i^1 s_k(\xi_0))_{k \in \mathbb{Z}}$ and $(\partial_i^2 s_k(\xi_0))_{k \in \mathbb{Z}}$ are in $l_1(\mathbb{Z})$. Moreover, $((\partial_1^1 s_k(\xi_0))_{k \in \mathbb{Z}}, \dots, (\partial_d^1 s_k(\xi_0))_{k \in \mathbb{Z}})$ and $(s_k(\xi_0))_{k \in \mathbb{Z}}$ are linearly independent.

Theorem 4.1. *Suppose that assumptions (M1)-(M7) hold, then as n goes to infinity, $\hat{\sigma}_0$ converges a.s. to σ_0 and $\|\hat{\xi}_0 - \xi_0\|$ converges a.s. to 0, where $\|\cdot\|$ denotes the Euclidean norm in \mathbb{R}^d .*

Corollary 4.1. *Suppose that assumptions (M1b), (M2)-(M7) hold, then as n goes to infinity, both $\|\hat{a} - a\|$ and $\|\hat{\Pi} - \Pi\|$ converge a.s. to 0, where $\|\cdot\|$ denotes the Euclidean norm in \mathbb{R}^p .*

Before giving the asymptotic distribution of our estimates, recall that the vector $v^* \in \mathbb{C}^{p+1}$ is the eigenvector associated with the smallest eigenvalue of $\tilde{D}(\sigma_0, \theta)$, and denote $v(b)$ any vector in $\mathbb{C}^{p+1} \cap \{\|\cdot\|_2 = 1\}$ associated with $b \in \mathbb{C}^p$ such that the $v(b)_j$'s are the complex coefficients of the polynomial function in $\mathbb{C}[X]$ having the components of b as roots. In particular, note that $v(a) = v^*$.

Theorem 4.2. *Under assumptions (M1)-(M8) and (P),*

$$\sqrt{n} \begin{pmatrix} \hat{\xi}_0 - \xi \\ \hat{\sigma}_0 - \sigma_0 \end{pmatrix} \xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N} \left(0_{d+1}, N D^1 h(\tilde{d}(\sigma_0, \xi_0)) M (D^1 h(\tilde{d}(\sigma_0, \xi_0)))' N' \right),$$

where 0_{d+1} is the $(d+1)$ -dimensional zero vector and

$$\begin{aligned} M &= A^{-1}(\sigma_0 \|\theta\|_2) \Gamma_1(A^{-1}(\sigma_0 \|\theta\|_2))', \\ N &= \frac{1}{\alpha} \begin{pmatrix} \partial_2^2 J(\sigma_0, \xi_0)^{-1} & \partial_{1,2}^{1,1} J(\sigma_0, \xi_0) \\ & 1 \end{pmatrix}, \\ \alpha &= -\partial_1^1 J(\sigma_0, \xi_0) \\ h(\tilde{d}(\sigma_0, \xi_0)) &= \det \left(\left(\tilde{d}_{j(p+1)+i+1}(\sigma_0, \xi_0) \right)_{i,j \in \{0, \dots, p\}} \right). \end{aligned}$$

Corollary 4.2. *Under (M1b), (M2)-(M8) and (P) and using the previous notations, one gets*

$$\begin{aligned} \sqrt{n}(\hat{a} - a) &\xrightarrow[n \rightarrow +\infty]{\mathcal{L}} \mathcal{N} \left(0_p, \frac{1}{4|v_p^*|^4} C^{-1} B R M \bar{B}' \bar{R}' C^{-1} \right), \\ \sqrt{n}(\hat{\Pi} - \Pi) &\xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N} \left(0_p, G R M \bar{R}' \bar{G}' \right), \end{aligned}$$

where,

$$\begin{aligned}
C &= \text{diag}(K_1, \dots, K_p), \text{ with } K_j = \mathbb{E} \left(\prod_{i=1, i \neq j}^p |X_0 - a_i|^2 \right), \\
B &\text{ is a } p \times (p+1)^2\text{-matrix which is defined by its columns as follows} \\
&\forall l \in \{1, \dots, p\}, \text{ with } (i, j) \in \{0, \dots, p\}^2, \\
&\quad B_{l, i(p+1)+j+1} = \left(\overline{\partial_l^1 v_i(a)} v_j^* + \partial_l^1 v_j(a) \overline{v_i^*} \right) \\
R &= \left(\text{Id}_{(p+1)^2} + \partial_{1,2}^{1,1} \tilde{d}(\sigma_0, \xi_0) D^1 h(\tilde{d}(\sigma_0, \xi_0)) N \right) \\
&\text{where } \text{Id}_d \text{ is the identity matrix of size } d, \\
G &= L^{-1} \left(\text{Proj} + F \frac{C^{-1}}{2|v_p^*|^2} B \right), \\
L &= (a_j^i)_{0 \leq i \leq p-1; 1 \leq j \leq p}, \text{ where } i \text{ denotes the rows and } j \text{ the columns,} \\
F &= \begin{pmatrix} 0_p' & (\pi_j \ i \ a_j^{i-1})_{i=2, \dots, p, j=1, \dots, p} \end{pmatrix}', \\
&\text{where } i \text{ denotes the rows, } j \text{ denotes the columns,} \\
\text{Proj} &\text{ is the projection of } \mathbb{C}^{(p+1)^2} \text{ on } \mathbb{C}^p \text{ such that} \\
&\forall w \in \mathbb{C}^{(p+1)^2}, \text{ Proj}(w) = v, \ v = (w_1, \dots, w_p)'.
\end{aligned}$$

Remark 4.1. *The proof of Theorem 4.2 is obtained using similar arguments as in Gassiat & Gautherat's proof of Theorem 4.2 [1999]. The gain, we obtain, with our estimation procedures (without penalty term) is the asymptotic marginal distribution of both $(\hat{\xi}_0 - \xi_0)$ and $(\hat{\sigma}_0 - \sigma_0)$. This is an essential point to obtain the asymptotic distribution of both \hat{a} and $\hat{\Pi}$.*

5. Simulation study

In this section, the estimates of σ_0 , ξ_0 , $(a_i)_{1 \leq i \leq p}$ and $(\pi_i)_{1 \leq i \leq p}$ are provided using our theoretical estimation procedures. To give stable results, each estimation value is an average over $N = 100$ independent simulations of sequence of n observations. These estimation values are denoted by \bar{E} in the arrays below. The stability of each estimation value is measured by "std", the empirical standard deviation calculated over the simulation runs. The simulations which lead to a negative value of $\hat{\Pi}$ are eliminated and N_{elim} in the arrays below is the number of those eliminated simulations.

Two models are considered in this simulation study: the mixture model and the autoregressive model that is, each simulation sequence $((Y_{t,l})_{1 \leq t \leq n, 1 \leq l \leq N})$ is issued of one of these models which are particular cases of model defined by (1.1). In both cases, the filter has a finite length, so we identify the inverse filter $\theta = s(\xi_0)$ to ξ_0 . We restrict ourselves to $p = 3$ and we deal with two values of σ_0 : $\sigma_0 = 0.05$ and $\sigma_0 = 1$; a small σ_0 (respectively a large σ_0) give

small perturbations (resp. large perturbations) to the corresponding non-noisy model defined by (1.1) so that it is more complicated to estimate it well in the case of a large σ_0 . To illustrate the asymptotic efficiency of our method, we deal with several n ($n = 50, 100, 250, 500, 750, 1000, 1500, 2000$) and $k(n)$ ($k(n) = 1, 2$) but only few significant results are presented here. We use the function "fsolve" in MATLAB version 7 which allows to find a root of a given function. The problem of the use of such a function is its very sensitivity to the starting point since it searches a zero near the starting point. To overcome this difficulty, we try several starting points in order to select as the initial point the one which seems more stable during the simulations. To avoid problems related on both scale and delay of θ , we fix the scale and delay in considering $\hat{\theta} = (\hat{\theta}_k)_{|k| \leq k(n)}$ such that $\|\hat{\theta}\|_{2,n} = 1$ and $(\hat{\theta}_{-k(n)})$ has the greatest value among all the components of $\hat{\theta}$.

Mixture model. The observed sequence Y_1, \dots, Y_n is given by

$$Y_t = X_t + \sigma_0 W_t, \quad \forall t \in \{1, \dots, n\},$$

so that the filter and its inverse coincide i.e. $\theta_t = u_t = \delta_0(t) \forall t \in \mathbb{Z}$. One must note that the model is over-parameterized since we choose $k(n) = 1$ (first array) and $k(n) = 2$ (second array) whereas the true inverse filter is reduced to one value $\theta_0 = 1$ which corresponds to $k(n) = 0$. The cases $n = 50$ and $n = 2000$ are presented in the arrays below. When $\sigma_0 = 0.05$, the estimation values of $(a_i)_{1 \leq i \leq 3}$ and $(\pi_i)_{1 \leq i \leq 3}$ are really good even for a small n ($n = 50$) and $k(n) = 2$. For both $\sigma_0 = 0.05$ and $\sigma_0 = 1$ and both $k(n) = 1$ and $k(n) = 2$, the estimation values of σ_0 and θ are strongly different between $n = 50$ and $n = 2000$: they are much more better for n large. These arrays illustrate that it is more difficult to obtain a good estimation when σ is large or/and when $k(n) = 2$ or/and when $n = 50$. It is worthwhile to note that for $\sigma_0 = 1$ the number of non-used sequences is $N_{elim} = 30$ ($n = 50, k(n) = 1$), $N_{elim} = 53$ ($n = 2000, k(n) = 1$), $N_{elim} = 38$ ($n = 50, k(n) = 2$) and $N_{elim} = 45$ ($n = 2000, k(n) = 2$), so that it would probably mean that we do not find an interesting starting point; moreover it entails a strong variability on the values (see the *std*). Figure 1 represents the n observations Y_1, \dots, Y_n , the support points a_1, a_2, a_3 and their estimates \hat{a}_1, \hat{a}_2 and \hat{a}_3 which are denoted $a_{i,esti}$, $i = 1, 2, 3$. When σ_0 is small ($\sigma_0 = 0.05$), the observations are concentrated on the true support points and the estimation for $k(n) = 1$ is visually very good even for a small sample (Figure 1, left side). When σ_0 is large ($\sigma_0 = 1$), the observations are more scattered over the square even one can distinguish three attractive areas where the support points lie. For this case and $k(n) = 1$, the estimation with 1000 observations is visually much more better than the one with 50 observations (Figure 1, right side). The same phenomenon is observed in Figure 2 but in addition Figure 2 illustrates the improvement of the estimation: when n is increasing, the estimates approach the true values and their empirical standard deviations tend to zero (except for one case).

Second order autoregressive model. The observed sequence Y_1, \dots, Y_n is

given by

$$Y_t = \tilde{Y}_t + \sigma_0 W_t, \forall t \in \{1, \dots, n\},$$

where

$$\tilde{Y}_t = \sum_{k=-\infty}^0 u_k X_{t-k} \iff X_t = \sum_{k=0}^2 \theta_k \tilde{Y}_{t-k},$$

with $(\theta_0, \theta_1, \theta_2) = (\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}) = (0.8571, -0.2857, 0.4286)$. One must note that when we choose $k(n) = 2$, the model is over-parameterized since the true model corresponds to $k(n) = 1$. In the left hand side of the third table ($k(n) = 1$) with a small $\sigma_0 = 0.05$, the estimations of all the parameters are quite good and they are quite similar for both large n and small n . All other cases i.e $\sigma_0 = 1$ with $k(n) = 1$ and $k(n) = 2$, large n ($n = 1000$ or $n = 2000$) lead to an improvement in the estimation values since some estimations for $n = 50$ are very far from the true values (see both the right hand side of the third table and the fourth table). However, in some cases it seems that we do not take an interesting starting point since the variability of the results is too large (see for example the *std* for $k(n) = 1$ and $\sigma_0 = 1$) combined with an important number of eliminated simulations (see N_{elim} for $k(n) = 1$ and $\sigma_0 = 1$). One can see in Figure 3 that the observations are more dispersed over the square than those of the mixture model (Figure 1). It would mean that an autoregressive model is more difficult to estimate than a mixture model. When σ_0 is small ($\sigma_0 = 0.05$), Figure 3 and Figure 4 show that the support's points $(a_i)_{1 \leq i \leq 3}$ are well estimated for $k(n) = 1$ even for a small n ; this is not the case for $k(n) = 2$ since the estimates seem visually (see Figure 4) far from the true values even for large n ($n = 1000$).

Numerical illustration of the existence of the estimate $\hat{\sigma}_0$. Let us consider the second order autoregressive model with $\sigma_0 = 1$ and let $k(n)$ be equal to 2. This corresponds to the most difficult estimation problem as it is illustrated in the above simulations. Figure 5 represents the graph of the function $\sigma \rightarrow G_{n, \bar{s}(\xi)}(\sigma) = \text{sign}(J_n(\sigma, \xi)) * \log(|J_n(\sigma, \xi)| + 1)$ which has the same roots as the function $\sigma \rightarrow J_n(\sigma, \xi)$. In the left hand side, we consider the particular case of $\bar{s}(\xi_0) = (\theta_0, \theta_1, \theta_2, 0, 0)$ which is the true value of the inverse filter, whereas in the right hand side, we consider the particular case of $\bar{s}(\xi) = (1, 0, 0, 0, 0)$ which differs from the true value of the inverse filter (this filter corresponds to a mixture model). In both cases, one can see that $G_{n, \bar{s}(\xi)}$ admits some zeros. In the left hand side, one must note that the convergence is achieved very quickly and accurately.

Importance of the choice of the starting point in the algorithm. One can see in Figure 5 that the first zero of the function J_n is neither over σ_0 nor under σ_0 . Actually, the Matlab toolbox algorithm searches the zero of the function which is the nearest to the starting point. This starting point being a $(2k(n)+2)$ -dimensional vector, the algorithm searches a zero in $(2k(n)+2)$ directions. So an iterative stochastic algorithm, which is able to find a multidimensional zero, would be an useful tool since the gradient of J_n could be formally computed.

$\sigma_0 = 0.05$ $k(n) = 1$	n	\hat{E}	$\pm std$	$\sigma_0 = 1$ $k(n) = 1$	n	\hat{E}	$\pm std$
σ_0	50 2000	0.1397 0.0554	± 0.2125 ± 0.0114	σ_0	50 2000	1.0139 1.1403	± 0.1672 ± 0.1936
θ_0	50 2000	0.9627 0.9999	± 0.0962 ± 0.0002	θ_0	50 2000	0.8821 0.9107	± 0.1100 ± 0.1296
θ_1	50 2000	0.0732 0.0000	± 0.1866 ± 0.0128	θ_1	50 2000	0.1192 0.0931	± 0.3169 ± 0.2834
θ_2	50 2000	0.0510 0.0006	± 0.1479 ± 0.0063	θ_2	50 2000	0.1087 0.0489	± 0.2935 ± 0.2566
a_3	50 2000	-2.1146 - 1.0012 i -2.0004 - 0.9996 i	± 0.2996 ± 0.0130	a_3	50 2000	-2.5278- 0.8964 i -2.1064 - 0.9275 i	± 1.3044 ± 0.2689
a_2	50 2000	-1.0265 + 2.9403 i -1.0002 + 3.0000 i	± 0.4038 ± 0.0129	a_2	50 2000	-1.1398 + 2.2255 i -0.7987 + 2.6327 i	± 1.6165 ± 0.8427
a_1	50 2000	3.6058 + 1.0336 i 3.9984 + 1.0000 i	± 1.1026 ± 0.0152	a_1	50 2000	2.9865+ 1.0596 i 3.7181 + 1.1201 i	± 1.6399 ± 0.5812
Π_1	50 2000	0.5906 0.6000	± 0.0690 ± 0.0104	Π_1	50 2000	0.4853 0.5681	± 0.1214 ± 0.0865
Π_2	50 2000	0.2468 0.2509	± 0.0671 ± 0.0091	Π_2	50 2000	0.3374 0.2923	± 0.1397 ± 0.0939
Π_3	50 2000	0.1625 0.1492	± 0.0597 ± 0.0078	Π_3	50 2000	0.1773 0.1396	± 0.0686 ± 0.0233
N_{elim}	50 2000	4 0		N_{elim}	50 2000	30 53	

TABLE 1

$(\theta_0, \theta_1, \theta_2, a_1, a_2, a_3, \pi_1, \pi_2, \pi_3) = (1, 0, 0, 4 + i, -1 + 3i, -2 - i, 0.6, 0.25, 0.15)$. The Starting point of left array is $(\sigma_{ini}, \theta_{ini,0}, \theta_{ini,1}, \theta_{ini,2}) = (0.001, 1.2, -0.5, 0.02)$ and the starting point of right array is $(\sigma_{ini}, \theta_{ini,0}, \theta_{ini,1}, \theta_{ini,2}) = (0.1, 1.2, -0.4, 0.2)$.

$\sigma_0 = 0.05$ $k(n) = 2$	n	\hat{E}	$\pm std$	$\sigma_0 = 1$ $k(n) = 2$	n	\hat{E}	$\pm std$
σ_0	50 2000	0.3913 0.0542	± 0.3550 ± 0.0078	σ_0	50 2000	1.0467 1.1649	± 0.1906 ± 0.2807
θ_0	50 2000	0.8362 0.9999	± 0.1686 ± 0.0001	θ_0	50 2000	0.7038 0.9195	± 0.1283 ± 0.1405
θ_1	50 2000	0.0674 -0.0003	± 0.2417 ± 0.0053	θ_1	50 2000	0.1393 -0.0087	± 0.3311 ± 0.1796
θ_2	50 2000	0.0840 0.0001	± 0.2086 ± 0.0080	θ_2	50 2000	0.1962 0.0232	± 0.2900 ± 0.1728
θ_3	50 2000	0.1308 0.0000	± 0.2520 ± 0.0047	θ_3	50 2000	0.1724 0.0077	± 0.2707 ± 0.2007
θ_4	50 2000	0.1026 -0.0000	± 0.2664 ± 0.0054	θ_4	50 2000	0.0998 0.0077	± 0.3609 ± 0.1858
a_3	50 2000	-2.4801 - 0.9127 i -1.9996 - 1.0002 i	± 0.9610 ± 0.0112	a_3	50 2000	-2.8623- 0.8196 i -2.0936 - 0.9478 i	± 1.5551 ± 0.3934
a_2	50 2000	-1.2051 + 2.5834 i -0.9997 + 3.0000 i	± 1.3587 ± 0.0115	a_2	50 2000	-1.0714 + 1.4250 i -0.7886 + 2.5256 i	± 2.2831 ± 0.9606
a_1	50 2000	3.1553 + 1.2781 i 3.9999 + 1.0001 i	± 1.5967 ± 0.0109	a_1	50 2000	2.4786 + 1.1597 i 3.7623 + 1.0584 i	± 2.3074 ± 0.8220
Π_1	50 2000	0.5211 0.5997	± 0.1270 ± 0.0115	Π_1	50 2000	0.4341 0.5436	± 0.1474 ± 0.1302
Π_2	50 2000	0.3011 0.2510	± 0.1314 ± 0.0091	Π_2	50 2000	0.3476 0.3161	± 0.1407 ± 0.1294
Π_3	50 2000	0.1778 0.1493	± 0.0812 ± 0.0084	Π_3	50 2000	0.2183 0.1403	± 0.1353 ± 0.0317
N_{elim}	50 2000	20 0		N_{elim}	50 2000	38 45	

TABLE 2

$(1, 0, 0, 4 + i, -1 + 3i, -2 - i, 0.6, 0.25, 0.15)$. Starting point of left array $(\sigma_{ini}, (\theta_{ini,i})_{0 \leq i \leq 4}) = (0.01, 0.3, -0.1, 0.2, -0.1, 0.1)$ and Starting point right array $(\sigma_{ini}, (\theta_{ini,i})_{0 \leq i \leq 4}) = (0.1, 0.3, -0.1, 0.2, -0.1, 0.1)$.

$\sigma_0 = 0.05$ $k(n) = 1$	n	\hat{E}	$\pm std$	$\sigma_0 = 1$ $k(n) = 1$	n	\hat{E}	$\pm std$
$\hat{\sigma}_0$	50 1000	0.0565 0.0591	± 0.0133 ± 0.0161	$\hat{\sigma}_0$	100 1000	1.1764 1.2805	± 0.2959 ± 0.3006
$\hat{\theta}_0$	50 1000	0.8627 0.8636	± 0.0045 ± 0.0068	$\hat{\theta}_0$	100 1000	0.8263 0.7814	± 0.0572 ± 0.0541
$\hat{\theta}_1$	50 1000	-0.2786 -0.2776	± 0.0076 ± 0.0098	$\hat{\theta}_1$	100 1000	0.0325 -0.1880	± 0.3910 ± 0.2438
$\hat{\theta}_2$	50 1000	0.4220 0.4207	± 0.0051 ± 0.0078	$\hat{\theta}_2$	100 1000	0.1640 0.4816	± 0.3700 ± 0.2503
\hat{a}_3	50 1000	-2.0130 - 1.0067i -2.0174 - 1.0095i	± 0.0182 ± 0.0240	\hat{a}_3	100 1000	-3.4094 - 0.8583i -2.3470 - 1.2209i	± 3.4494 ± 1.1879
\hat{a}_2	50 1000	-1.0090 + 3.0197i -1.0088 + 3.0246i	± 0.0276 ± 0.0319	\hat{a}_2	100 1000	-1.9715 + 3.5025i -1.0180 + 2.7762i	± 3.6024 ± 2.0138
\hat{a}_1	50 1000	4.0261 + 1.0081i 4.0338 + 1.0082i	± 0.0299 ± 0.0395	\hat{a}_1	100 1000	5.3438 + 1.1887i 3.4259 + 1.0842 i	± 5.2553 ± 1.8686
$\hat{\Pi}_1$	50 1000	0.6087 0.5998	± 0.0539 ± 0.0147	$\hat{\Pi}_1$	100 1000	0.5078 0.5235	± 0.1227 ± 0.1352
$\hat{\Pi}_2$	50 1000	0.2517 0.2504	± 0.0489 ± 0.0133	$\hat{\Pi}_2$	100 1000	0.3075 0.2877	± 0.0898 ± 0.0833
$\hat{\Pi}_3$	50 1000	0.1396 0.1498	± 0.0326 ± 0.0105	$\hat{\Pi}_3$	100 1000	0.1847 0.1887	± 0.0635 ± 0.1088
N_{elim}	50 1000	0 0		N_{elim}	100 1000	58 56	

TABLE 3

$(\theta_0, \theta_1, \theta_2, a_1, a_2, a_3, \Pi_1, \Pi_2, \Pi_3) = (6/7, -2/7, 3/7, 4 + i, -1 + 3i, -2 - i, 0.6, 0.25, 0.15)$. The starting point of left array is $(\sigma_{ini}, \theta_{ini,0}, \theta_{ini,1}, \theta_{ini,2}) = (0.001, 0.5, -0.5, 0.5)$ whereas the right array's is $(\sigma_{ini}, \theta_{ini,0}, \theta_{ini,1}, \theta_{ini,2}) = (0.5, 0.6, -0.2, 0.2)$.

$\sigma_0 = 0.05$ $k(n) = 2$	n	E	$\pm std$	$\sigma_0 = 1$ $k(n) = 2$	n	E	$\pm std$
σ_0	50 1000	0.1642 0.0580	± 0.3229 ± 0.0080	σ_0	100 2000	1.1082 1.2655	± 0.1266 ± 0.0751
θ_0	50 1000	0.8090 0.8632	± 0.0770 ± 0.0010	θ_0	100 2000	0.8314 0.8390	± 0.0613 ± 0.0475
θ_1	50 1000	-0.2855 -0.2787	± 0.0695 ± 0.0019	θ_1	100 2000	-0.3027 -0.2961	± 0.1049 ± 0.1154
θ_2	50 1000	0.3815 0.4209	± 0.0754 ± 0.0011	θ_2	100 2000	0.2419 0.2415	± 0.1115 ± 0.1018
θ_3	50 1000	0.1552 0.0002	± 0.1866 ± 0.0018	θ_3	100 2000	0.1408 0.1062	± 0.1808 ± 0.1910
θ_4	50 1000	-0.1278 -0.0014	± 0.1658 ± 0.0014	θ_4	100 2000	-0.2608 -0.2639	± 0.1080 ± 0.0874
a_3	50 1000	-2.5866 - 0.6725i -2.0125 - 1.0080 i	± 2.5778 ± 0.0050	a_3	100 2000	-2.3404 - 1.1354i -1.9004 - 1.1264 i	± 1.3817 ± 0.2632
a_2	50 1000	-1.4271 + 2.8472 i -1.0046 + 3.0204 i	± 2.0162 ± 0.0068	a_2	100 2000	-1.0821 + 2.7644i -0.7037 + 2.5984 i	± 1.6653 ± 0.3441
a_1	50 1000	4.2549 + 1.4208 i 4.0316 + 1.0070 i	± 1.2687 ± 0.0085	a_1	100 2000	4.3995 + 1.0729 i 4.1748 + 0.9867 i	± 0.9172 ± 0.3380
Π_1	50 1000	0.5268 0.6000	± 0.1337 ± 0.0145	Π_1	100 2000	0.5133 0.5678	± 0.1108 ± 0.0451
Π_2	50 1000	0.2983 0.2508	± 0.1169 ± 0.0117	Π_2	100 2000	0.3115 0.2799	± 0.0991 ± 0.0363
Π_3	50 1000	0.1749 0.1492	± 0.0664 ± 0.0103	Π_3	100 2000	0.1752 0.1524	± 0.0544 ± 0.0142
N_{elim}	50 1000	3 0		N_{elim}	100 2000	11 14	

TABLE 4

$(\theta_0, \theta_1, \theta_2, a_1, a_2, a_3, \pi_1, \pi_2, \pi_3) = (6/7, -2/7, 3/7, 4 + i, -1 + 3i, -2 - i, 0.6, 0.25, 0.15)$. The starting point of left array is $(\sigma_{ini}, (\theta_{ini,i})_{0 \leq i \leq 4}) = (0.005, 0.2, 0.3, 0.2, 0.2, 0.2)$ and the starting point of right array is $(\sigma_{ini}, (\theta_{ini,i})_{0 \leq i \leq 4}) = (0.7, 0.2, -0.3, 0.2, 0.002, 0.001)$.

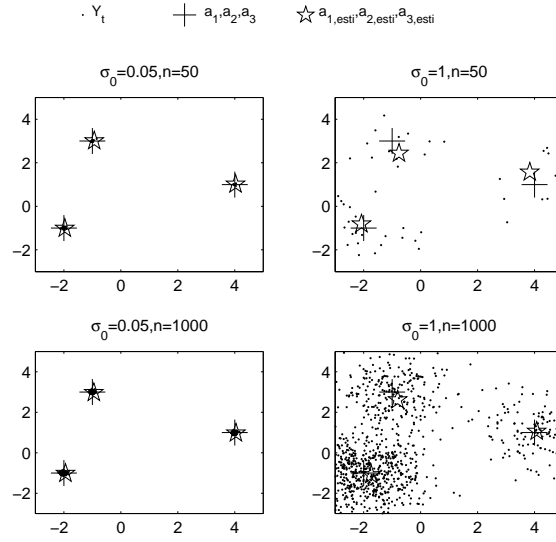


FIG 1. *Mixture Model, $k(n) = 1$.*

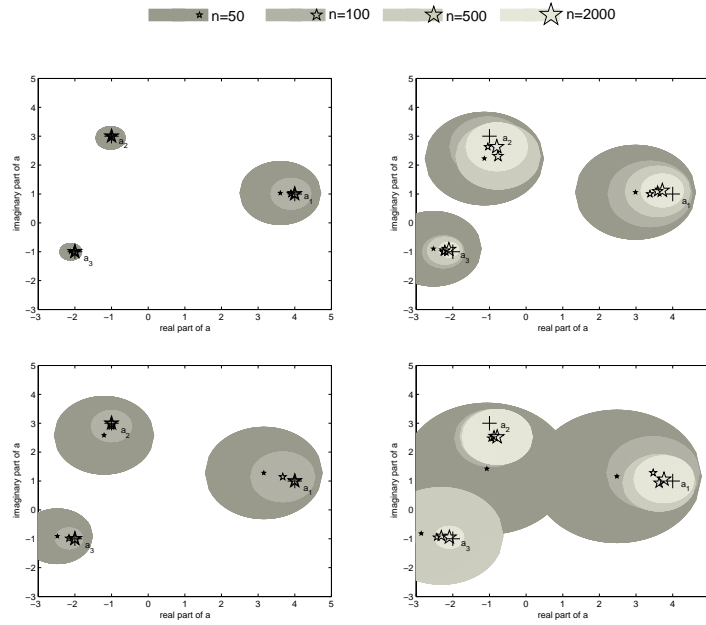


FIG 2. *Evolution of \hat{a}_1, \hat{a}_2 and \hat{a}_3 with n . On the first line $k(n) = 1$, on the second line $k(n) = 2$. For the first row $\sigma_0 = 0.05$ and for the second row $\sigma_0 = 1$.*

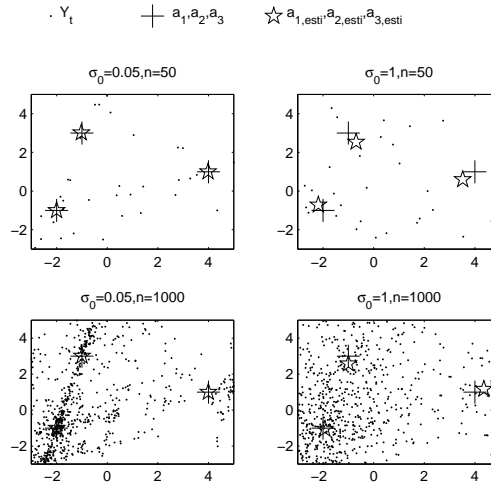


FIG 3. Second order autoregressive, $k(n) = 1$.

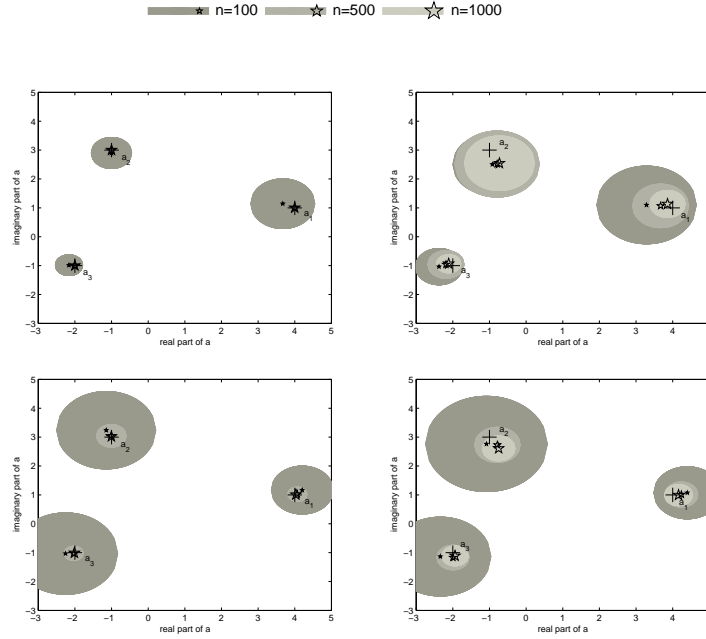


FIG 4. Evolution of \hat{a}_1, \hat{a}_2 and \hat{a}_3 with n . On the first line $k(n) = 1$, on the second line $k(n) = 2$. For the first row $\sigma_0 = 0.05$ and for the second row $\sigma_0 = 1$.

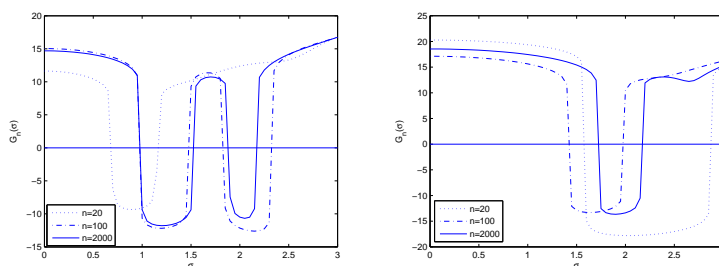


FIG 5. Graph of $G_{n, \bar{s}(\xi)}$ for $k(n) = 2$. Second order autoregressive, $\sigma_0 = 1$.

6. Discussion

- **Interest of our estimation procedure.** Our procedure estimation does not require a priori more specifications of the model than equation (1.1) then it takes the advantage to adapt to any situations.

- **Choice of the Hankel matrix.** It would be possible to deal with the Toeplitz matrix instead of the Hankel matrix of the (Z_t) since the characterizations of θ and σ_0 given by relations (2.7) and (2.6) also hold for the Toeplitz matrix. More generally for the same reason, it would be possible to consider any $(p+1)^2$ -vector built on the moment of type $(\mathbb{E}((\phi_1(Z_0(s(\xi))))^k \overline{\phi_1(Z_0(s(\xi)))^j}))_{k,j \in \{1, \dots, p\}}$ where ϕ_1 is any complex injective function defined on \mathbb{C} . The main difference would lie in the non-trivial determination of A^{-1} , the inverse matrix of A defined in relation (2.3). It also could be extend to some entropy distance which allows to distinguish variables which have less than p point of support than the others (Gamboa & Gassiat [1996]).

- **Gaussian noise.** The assumption of a Gaussian noise is not necessary; actually only an indivisible law is required for the noise. As previously, it is probably more complicated to exhibit the matrix A^{-1} which contains among others the calculations of the $\gamma_{j,k}$.

- **Method of moments.** Traditionally the method of the moments is not very well appreciated in estimation problems since a little error on the observations entails a big error in the final estimation. In our case, since σ_0 is not a priori estimated, our method takes intrinsically into account such a type of error and does not possess the disadvantage of the method of moments. Nevertheless, the restitution of the distribution of $(X_t)_{t \in \mathbb{Z}}$ is based on this method and it would be probably more efficient to apply the MCMC methods (see for example Sylvia [2001]) since the application of the inverse filter provides a finite mixture with a known covariance structure.

- **Computational comparison.** In order to compare empirically our method with already existing numerical results (Gassiat & Gautherat [1998], p. 1947), we consider the second order autoregressive model with a real signal, for $p = 2$

and $\theta = (\theta_0, \theta_1, \theta_2) = (\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}) = (0.8571, -0.2857, 0.4286)$.

All quantities which appear in our estimation procedures, are adapted to the real case (in particular the matrix A). We characterize the results obtained in Gassiat *et al.* [1998] by \mathcal{M}_1 and our results by \mathcal{M}_2 . The algorithm we implement here consists in selecting randomly a starting point for which the criteria function J_n takes a value close to zero, and then the pre-defined function `fsolve` in Matlab is used to find the zero. The random selection is made from points (θ_0, θ_1) which are uniformly distributed on the sphere S_1 . The signal-to-noise ratio expressed in DB is denoted SNR.

σ_0	SNR	n	méthode	$\widehat{E}(\theta_0) \pm std(\theta_0)$	$\widehat{E}(\theta_1) \pm std(\theta_1)$	$\widehat{E}(\sigma_0) \pm std(\sigma_0)$
0.1	46	100	\mathcal{M}_1	0.7500 ± 0.0647	-0.1767 ± 0.0668	0.0000 ± 0.0000
0.1	46	100	\mathcal{M}_2	0.8553 ± 0.0095	-0.2879 ± 0.0205	0.1004 ± 0.0100
0.1	46	500	\mathcal{M}_1	0.7591 ± 0.0140	-0.1860 ± 0.0154	0.0000 ± 0.0000
0.1	46	500	\mathcal{M}_2	0.8580 ± 0.0049	-0.2941 ± 0.0187	0.1035 ± 0.0031
0.1	46	1000	\mathcal{M}_1	0.7639 ± 0.0128	-0.1923 ± 0.0142	0.0000 ± 0.0000
0.1	46	1000	\mathcal{M}_2	0.8573 ± 0.0080	-0.2837 ± 0.0151	0.1009 ± 0.0139
1	0.46	500	\mathcal{M}_1	0.5913 ± 0.2537	-0.0916 ± 0.2652	0.9407 ± 0.1015
1	0.46	500	\mathcal{M}_2	0.1423 ± 0.6017	-0.4147 ± 0.2475	0.8607 ± 0.5653
1	0.46	5000	\mathcal{M}_1	0.7529 ± 0.1211	-0.2450 ± 0.1353	1.0022 ± 0.0612
1	0.46	5000	\mathcal{M}_2	0.5491 ± 0.5060	-0.3259 ± 0.2020	0.9646 ± 0.3915
1	0.46	15000	\mathcal{M}_1	0.7771 ± 0.1294	-0.2576 ± 0.1206	1.0302 ± 0.0651
1	0.46	15000	\mathcal{M}_2	0.7159 ± 0.3757	-0.2928 ± 0.1476	0.9844 ± 0.0387

One must note that for a large SNR, the method \mathcal{M}_2 is more performant than the method \mathcal{M}_1 both in the estimated values and in the empirical standard deviation. The non-asymptotic side of our method is particularly highlighted in the case of a large SNR: when SNR is equal to 46, our method works well even for a small sample size and it is always better than the method \mathcal{M}_1 . For a very small SNR (equal to 0.46), the situation is changing: no method is able to handle such a level of noise; to our knowledge, only the paper of Gassiat & Gautherat [1998] gave numerical results in this situation. One can note that, even a priori the results do not give satisfactory, our estimation methods works better and better with an increasing n .

Another aspect which is of importance, is that our method does not need to calibrate some parameters, on the contrary of the estimation method in Gassiat & Gautherat [1998], which is based on the minimization of a penalized contrast function. Moreover our method does not require a starting point that we have to fixed in advance since, as it is mentioned above, the starting point is selected randomly: it is not the case for the results of the section 5 which are sensitive to the starting point we choose nor for the numerical results of the method \mathcal{M}_1 described in Gassiat & Gautherat [1998]. The method \mathcal{M}_1 needs more: the starting point must be near enough to the true valeur, otherwise the method could provide a local minimum. The gain of our method is that no a priori on extra parameters and on a starting point is requested, but its drawback is that in some cases, it generates a large standard deviation. The outlook of the future use of the inverse matrix A^{-1} since it is explicit done, would probably perform the numerical results.

7. Proofs

For convenience, denote $J(\sigma, s(\xi)) = J(\sigma, \xi)$. We first give a very useful tool which is a combination of existing results obtained by Gautherat [1997] (see

Lemma 5.3.1 p.130) and Gassiat & Gautherat [1999] (see Lemma 4.1. p. 1695):

Lemma 7.1. *Under assumptions (M1)-(M8) and (P), one gets*

- i) $\forall \sigma \in \mathbb{R}_+^*, \forall \xi \in \mathcal{K}, J_n(\sigma, \xi) \xrightarrow[n \rightarrow \infty]{\mathbb{P}-a.s.} J(\sigma, \xi).$
- ii) $\forall n \in \mathbb{N}, \forall \xi \in \mathcal{K}, J_n(\sigma, \xi)$ is differentiable with respect to σ and $\partial_1^1 J_n(\sigma_0, \xi_0) \xrightarrow[n \rightarrow \infty]{\mathbb{P}-a.s.} \partial_1^1 J(\sigma_0, \xi_0) = -\alpha < 0.$
- iii) *The function $s(\xi) \in \Theta \mapsto \partial_1 J_n(\sigma_0, \bar{s}(\xi))$ is continuous on Θ .*
- iv) $\forall n \in \mathbb{N}, J_n(\sigma, \xi)$ is twice differentiable in (σ_0, ξ_0) with respect to both σ and ξ . The first and second derivatives of $J_n(\sigma, \xi)$ in σ_0 and ξ_0 converge $\mathbb{P} - a.s.$ to the first and second derivative of $J(\sigma, \xi)$ in (σ_0, ξ_0) .
- v) *The asymptotic distribution of $(\sqrt{n} \partial_2^1 J_n(\sigma_0, \xi_0), \sqrt{n} J_n(\sigma_0, \xi_0))$ is a centered Gaussian vector variance*

$$D^1 h(d(\sigma_0, \xi_0)) A^{-1}(\sigma_0 \|s(\xi_0)\|_2) \Gamma_1(A^{-1}(\sigma_0 \|s(\xi_0)\|_2))' (D^1 h(d(\sigma_0, \xi_0)))'.$$

Proof of Lemma 7.1 These results are proved using the compactness of \mathcal{K} and in adapting the proof of Lemma 4.1 in Gassiat & Gautherat [1999] to the almost surely convergence for **i)-iv)**, and directly from the statement adapted to the almost-surely convergence of Gassiat & Gautherat [1999], Lemma 4.1 for **ii)-v)**. Whereas, **iii)** is obtained due to the truncation $k(n)$ of $s(\xi)$ and the polynomial structure of J_n .

Proof of Theorem 4.1.

• **Consistency of $\hat{\sigma}_0$.** Let \mathcal{V}_{σ_0} be some neighborhood of σ_0 and choose $\sigma_1 < \sigma_0$ in \mathcal{V}_{σ_0} . Due to relation (2.6), σ_1 satisfies $J(\sigma_1, \xi) > 0$ for all ξ . On the other hand, due to assertion **ii)** of Lemma 7.1 it is always possible to consider $\sigma_2 > \sigma_0$ in \mathcal{V}_{σ_0} such that $J(\sigma_2, \xi_0) < 0$ and such that for any $\sigma \in]\sigma_0, \sigma_2[$, $J(\sigma, \xi_0) < 0$. Assertion **i)** in Lemma 7.1 leads to

$$\begin{aligned} J_n(\sigma_1, \xi_0) &\xrightarrow[n \rightarrow \infty]{\mathbb{P}-a.s.} J(\sigma_1, \xi_0), \\ J_n(\sigma_2, \xi_0) &\xrightarrow[n \rightarrow \infty]{\mathbb{P}-a.s.} J(\sigma_2, \xi_0). \end{aligned}$$

Let $0 < \epsilon < \inf\{J(\sigma_1, \xi_0), |J(\sigma_2, \xi_0)|\}$. Then, it exists a positive integer N_0 such that for all $n \geq N_0$, $J_n(\sigma_1, \xi_0) > 0$ and $J_n(\sigma_2, \xi_0) < 0$. Thus, from **ii)** in Lemma 7.1, it follows that for all $n > N_0$, it exists $\tilde{\sigma}_n \in]\sigma_1, \sigma_2[$ such that $J_n(\tilde{\sigma}_n, \xi_0) = 0$ and we choose $\tilde{\sigma}_n$ such that $\tilde{\sigma}_n = \inf\{\sigma_n \in]\sigma_1, \sigma_2[: J_n(\sigma_n, \xi_0) = 0\}$. From Assertions **i)** and **ii)** in Lemma 7.1, and a Taylor expansion of J_n at $(\tilde{\sigma}_n, \xi_0)$, one obtains,

$$J_n(\tilde{\sigma}_n, \xi_0) = J_n(\sigma_0, \xi_0) + (\tilde{\sigma}_n - \sigma_0) \partial_1^1 J_n(\sigma_0, \xi_0)(1 + o(1)),$$

and one gets $(\tilde{\sigma}_n - \sigma_0) \xrightarrow[n \rightarrow \infty]{\mathbb{P}-a.s.} 0$. Since for all $\sigma < \sigma_1$, $J_n(\sigma, \xi_0) > 0$, $\hat{\sigma}_0$ satisfies $\hat{\sigma}_0 > \sigma_1$. Consider only large n that is n such that $n > N_0$, by definition of $\hat{\sigma}_0$, one has $\hat{\sigma}_0 < \tilde{\sigma}_n$. Since we consider only σ lying in the compact set $[\sigma_1, \sigma_2]$,

there exists a subsequence $\tilde{\sigma}_{0,n}$ of $\hat{\sigma}_0$ which converges to $\tilde{\sigma}_0$ and which satisfies $\tilde{\sigma}_0 < \sigma_0$. Since $J_n(\tilde{\sigma}_{0,n}, \xi_0) = 0$ and due to $J_n(\tilde{\sigma}_{0,n}, \xi_0) \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} J(\tilde{\sigma}_0, \xi_0)$, it follows that $J(\tilde{\sigma}_0, \xi_0) = 0$, which contradicts the definition of σ_0 (see relation (2.6)). This achieves the proof.

• **Consistency of $\hat{\xi}_0$.** Consider only σ 's in $[\sigma_1, \sigma_2]$. Since \mathcal{K} is a compact set, $\hat{\xi}_0$ admits a subsequence $(\tilde{\xi}_n)_n$ which converges to $\tilde{\xi}_0$. Assertion i) in Lemma 7.1, the a.s.-convergence of $\hat{\sigma}_0$ and the continuity of J_n and s lead to $J_n(\hat{\sigma}_0, \tilde{\xi}_n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} J(\sigma_0, \tilde{\xi}_0)$. This implies that $\tilde{\xi}_0$ is equal to ξ_0 since $J(\sigma_0, \tilde{\xi}_0) = 0 \iff s(\tilde{\xi}_0) = \theta$. Suppose now there exists $\bar{\xi}_0$ an accumulation point which is different from ξ_0 . Then it exists another subsequence $\bar{\xi}_n$ of $\hat{\xi}_0$ which converges to $\bar{\xi}_0$. Using the same tricks as previously, one gets $s(\bar{\xi}_0) = \theta$ which proves the uniqueness of $\tilde{\xi}_0$.

Proof of Corollary 4.1. This proof is explicitly done in Gautherat [2002] (see proof of Theorem 3.2). It is only based on the consistency of $\hat{\sigma}_0$ and $\hat{\xi}_0$.

Proof of Theorem 4.2. The definition of $\hat{\sigma}_0$ leads to $J_n(\hat{\sigma}_0, \hat{\xi}_0) = 0$. It entails that both $\partial_1^1 J_n^2(\hat{\sigma}_0, \hat{\xi}_0) = 0$ and $\partial_2^1 J_n^2(\hat{\sigma}_0, \hat{\xi}_0) = 0_d$. For simplicity's sake, denote $J_n(\sigma_0, \xi_0) = J_n$, $\partial_i^{r_i} J_n(\sigma_0, \xi_0) = \partial_i^{r_i} J_n$, $i = 1, 2$ and $\partial_{i,j}^{r_i, r_j} J_n(\sigma_0, \xi_0) = \partial_{i,j}^{r_i, r_j} J_n$, $i, j = 1, 2$. Therefore, one can apply the Delta method to J_n^2 at (σ_0, ξ_0) , since

$$\begin{pmatrix} \partial_2^1 J_n^2(\hat{\sigma}_0, \hat{\xi}_0) \\ \partial_1^1 J_n^2(\hat{\sigma}_0, \hat{\xi}_0) \end{pmatrix} = \begin{pmatrix} 2J_n(\hat{\sigma}_0, \hat{\xi}_0) \partial_2^1 J_n(\hat{\sigma}_0, \hat{\xi}_0) \\ 2J_n(\hat{\sigma}_0, \hat{\xi}_0) \partial_1^1 J_n(\hat{\sigma}_0, \hat{\xi}_0) \end{pmatrix} = \begin{pmatrix} 0_d \\ 0 \end{pmatrix}.$$

Now, the expansion at the first order of J_n^2 at (σ_0, ξ_0) is

$$\begin{pmatrix} 2(\partial_2^1 J_n)' \partial_2^1 J_n + 2J_n \partial_2^2 J_n & 2(\partial_2^1 J_n)' \partial_1^1 J_n + 2J_n \partial_{1,2}^{1,1} J_n \\ 2\partial_2^1 J_n \partial_1^1 J_n + 2J_n (\partial_{1,2}^{1,1} J_n)' & 2(\partial_1^1 J_n)^2 + 2J_n D^1 J_n \end{pmatrix} \begin{pmatrix} \hat{\xi}_0 - \xi_0 \\ \hat{\sigma}_0 - \sigma_0 \end{pmatrix} (1 + o(1)) + \begin{pmatrix} 2J_n \partial_2^1 J_n \\ 2J_n \partial_1^1 J_n \end{pmatrix} = \begin{pmatrix} 0_d \\ 0 \end{pmatrix}$$

Denote $A_n = (\partial_2^1 J_n)' \partial_2^1 J_n + J_n \partial_2^2 J_n$, $B_n = (\partial_2^1 J_n)' \partial_1^1 J_n + J_n \partial_{1,2}^{1,1} J_n$ and $d_n = (\partial_1^1 J_n)^2 + J_n \partial_1^2 J_n$. Then, from the Schur complement (Searle [1982]), we obtain

$$\begin{aligned} & \begin{pmatrix} \hat{\xi}_0 - \xi_0 \\ \hat{\sigma}_0 - \sigma_0 \end{pmatrix} (1 + o(1)) \\ &= \\ & \begin{pmatrix} J_n A_n^{-1} (\partial_2^1 J_n)' + \frac{J_n}{d_n - B_n' A_n^{-1} B_n} (A_n^{-1} B_n B_n' A_n^{-1} (\partial_2^1 J_n)' - A_n^{-1} B_n \partial_1^1 J_n) \\ \frac{J_n}{d_n - B_n' A_n^{-1} B_n} (-B_n' A_n^{-1} (\partial_2^1 J_n)' + \partial_1^1 J_n) \end{pmatrix}. \end{aligned}$$

For n fixed, note that $J_n(\sigma_0, \xi_0)$ differs from zero. Thus, one could rewrite the up-right term in the previous equation using the expression of B_n and dividing

it by J_n ; it gives three terms T_1 , T_2 and T_3 which are defined by:

$$\begin{aligned} T_1 &= \left(\frac{A_n}{J_n} \right)^{-1} \left(1 - \frac{(\partial_1^1 J_n)^2}{d_n - B_n' A_n^{-1} B_n} \right) (\partial_2^1 J_n)', \\ T_2 &= \left(\frac{A_n}{J_n} \right)^{-1} \frac{B_n B_n' A_n^{-1} (\partial_2^1 J_n)'}{d_n - B_n' A_n^{-1} B_n}, \\ T_3 &= - \left(\frac{A_n}{J_n} \right)^{-1} \frac{\partial_1^1 J_n}{d_n - B_n' A_n^{-1} B_n} J_n \partial_{1,2}^{1,1} J_n. \end{aligned}$$

Rewrite the approximation of the vector $(\hat{\xi}_0 - \xi_0, \hat{\sigma}_0 - \sigma_0)'$ as follows

$$J_n \left(\frac{1}{d_n - B_n' A_n^{-1} B_n} (T_1 + T_2 + T_3) \right).$$

Due to Lemma 7.1 and the continuity of J_n in ξ , one has $A_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} 0$, $B_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} 0$, $d_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} (\partial_1^1 J(\sigma_0, \xi_0))^2 = \alpha^2 > 0$, $\frac{A_n}{J_n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \partial_2^2 J(\sigma_0, \xi_0)$, $\frac{B_n}{J_n} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} W$, where W is a d -dimensional non degenerate random vector and $\frac{d_n}{J_n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} \pm\infty$. As n large enough, it entails that

$$\sqrt{n} \begin{pmatrix} \hat{\xi}_0 - \xi_0 \\ \hat{\sigma}_0 - \sigma_0 \end{pmatrix} \stackrel{\mathcal{L}}{=} \sqrt{n} J_n(\sigma_0, \xi_0) \begin{pmatrix} \frac{(\partial_2^2 J(\sigma_0, \xi_0))^{-1} (\partial_{1,2}^{1,1} J(\sigma_0, \xi_0))}{\alpha} \\ \frac{1}{\alpha} \end{pmatrix} \quad (7.1)$$

Note that $h(\tilde{d}(\sigma_0, \xi_0)) = J_n(\sigma_0, \xi_0)$, where h is the determinant function. Then, due to Assumption (M7) and due to the Taylor expansion of h at $\tilde{d}(\sigma_0, \xi_0)$, one obtains

$$h(\tilde{d}(\sigma_0, \xi_0)) = D^1 h(\tilde{d}(\sigma_0, \xi_0)) A^{-1}(\sigma_0, \|\theta\|_2) (d_n(\xi_0) - d(\xi_0) + o(\frac{1}{\sqrt{n}})) \quad (7.2)$$

Set $M = A^{-1}(\sigma_0, \|s(\xi_0)\|_2) \Gamma_1(A^{-1}(\sigma_0, \|s(\xi_0)\|_2))'$ and

$$N = \begin{pmatrix} \frac{1}{\alpha} (\partial_2^2 J(\sigma_0, \xi_0))^{-1} \partial_{1,2}^{1,1} J(\sigma_0, \xi_0) \\ \frac{1}{\alpha} \end{pmatrix}, \text{ then due to (7.1), (7.2) and } \mathbf{v} \text{ of}$$

Lemma 7.1 one gets,

$$\sqrt{n} \begin{pmatrix} \hat{\xi}_0 - \xi_0 \\ \hat{\sigma}_0 - \sigma_0 \end{pmatrix} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N} \left(0_{d+1}, N D^1 h(\tilde{d}(\sigma_0, \xi_0)) M (D^1 h(\tilde{d}(\sigma_0, \xi_0)))' N' \right).$$

Proof of Corollary 4.2. Following the proof of Theorem 3.3 in Gautherat [2002], it remains to obtain an equivalent for $\sqrt{n}(\tilde{d}_n(\hat{\sigma}_0, \hat{\xi}_0) - \tilde{d}(\sigma_0, \xi_0))$. As n large enough, this term is equivalent in distribution to

$$\left(\text{Id}_{(p+1)^2} + D^1 \tilde{d}(\sigma_0, \xi_0) D^1 h(d(\sigma_0, \xi_0)) N \right) A^{-1}(\sigma_0, \|\theta\|_2) \sqrt{n} (d_n(\xi_0) - d(\xi_0)).$$

On the other hand, one has

$$\sqrt{n}(\hat{a} - a) = \left(\frac{C^{-1}}{2|v_p^*|^2} B \right) \sqrt{n}(\tilde{d}_n(\hat{\sigma}_0, \hat{\xi}_0) - \tilde{d}(\sigma_0, \xi_0)), \quad (7.3)$$

$$\sqrt{n}(\hat{\Pi} - \Pi) = L^{-1} \left(Proj + F \frac{C^{-1}}{2|v_p^*|^2} B \right) \sqrt{n}(\tilde{d}_n(\hat{\sigma}_0, \hat{\xi}_0) - \tilde{d}(\sigma_0, \xi_0)). \quad (7.4)$$

Relations (7.3) and (7.4) entail the results. All matrices used here are defined in the statement of both Theorem 4.2 or Corollary 4.2.

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